A GEOMETRICAL PROPERTY OF $C(K)$ SPACES†

BY

D. E. ALSPACH[††,a] AND Y. BENYAMINI[‡,b]

[a]*Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-0613, USA; and*
[b]*Department of Mathematics, Technion — Israel Institute of Technology, Haifa 32000, Israel*

ABSTRACT

We introduce a geometrical property of norm one complemented subspaces of $C(K)$ spaces which is useful for computing lower bounds on the norms of projections onto subspaces of $C(K)$ spaces. Loosely speaking, in the dual of such a space if x^* is a w* limit of a net (x_α^*) and $x^* = x_1^* + x_2^*$ with $\| x^* \| = \| x_1^* \| + \| x_2^* \|$, then we measure how efficiently the x_α^*'s can be split into two nets converging to x_1^* and x_2^*, respectively. As applications of this idea we prove that if for every $\varepsilon > 0$, X is a norm $(1+\varepsilon)$ complemented subspace of a $C(K)$ space, then it is norm one complemented in some $C(K)$ space, and we give a simpler proof that a slight modification of an l_1-predual constructed by Benyamini and Lindenstrauss is not complemented in any $C(K)$ space.

0. Introduction

Much work has been done on complemented subspaces of $C(K)$ spaces. The main open problem is whether such a space is necessarily isomorphic to a $C(K)$ space. For lack of intrinsic invariants, most of the results were obtained as immediate specialization to projections of deep results on general operators on $C(K)$ spaces or by *ad hoc* computations. In this article we present a very simple geometric property of $C(K)$ spaces which is inherited by its norm one complemented subspaces. We believe that this property can be exploited further in the classification of the complemented subspaces of $C(K)$ spaces. As

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has often been the case in other work on the structure of $C(K)$ spaces, the property we use relates the w* and norm topologies of X^*. Proposition 0.1 describes the property we employ. (Convergence in the statement of the proposition below refers to the w* topology.)

PROPOSITION 0.1. *Let K be a compact Hausdorff space, and let $\mathcal{M}$ be a bounded subset of $C(K)^*$. Suppose that μ is a w*-limit point of $\mathcal{M}$ and that there are μ_1 and μ_2 in $C(K)^*$ such that $\mu = \mu_1 + \mu_2$ and $\|\mu_1\| + \|\mu_2\| = \|\mu\|$. Then there is a net (ν^α) in $\mathcal{M}$ and for each α elements ν_1^α, ν_2^α, and η^α of $C(K)^*$ such that*

(i) $\nu^\alpha = \nu_1^\alpha + \nu_2^\alpha + \eta^\alpha$,

(ii) $\nu^\alpha \to \mu$, $\nu_1^\alpha \to \mu_1$, $\nu_2^\alpha \to \mu_2$ *and* $\eta^\alpha \to 0$,

(iii) $\|\nu_i^\alpha\| \leqq \|\mu_i\|$ *for all* α *and* $i = 1$ *and* 2,

(iv) $\lim_\alpha(\|\nu^\alpha\| - \|\eta^\alpha\| - \|\mu\|) = 0$.

Note that because of (iii) the limit in (iv) cannot be positive. This means that the splitting $\nu^\alpha = \nu_1^\alpha + \nu_2^\alpha + \eta^\alpha$ is as "efficient" as possible in the sense that the triangle inequality becomes an equality in the limit. It turns out that for general spaces X the minimal defect in the triangle inequality for possible splittings in X^* can be used to give lower estimates on the norm of projections from $C(K)$ onto X. This is formalized in Theorem 1.3, and the rest of the paper is devoted to some applications of this idea.

Given a Banach space X we denote by $\lambda(X)$ its projection constant relative to $C(K)$ spaces, i.e., $\lambda(X) = \inf \|T\| \, \|T^{-1}\| \, \|P\|$, where the infimum is over all possible isomorphisms T from X into a $C(K)$ space and projections P from $C(K)$ onto TX. By the Lemma in [B–L] actually $\lambda(X) = \inf \|P\|$, where the infimum is over projections P from a $C(K)$ space onto a subspace isometric to X. If X is separable, it is sufficient to consider metrizable K in these definitions.

In Section 2 we prove the following stability result for spaces with $\lambda(X) = 1$.

THEOREM 0.2. *Let X be a Banach space with $\lambda(X) = 1$. Then X is isometric to a norm one complemented subspace of some $C(K)$ space.*

Theorem 0.2 generalizes [B, Theorem 1] which says that a space which is almost isometric to a norm one complemented subspace Y of a $C(K)$ space is isometric to Y. Indeed by Theorem 0.2 such a space is isometric to a norm one complemented subspace of a $C(K)$ space, and two such spaces are isometric if they are isomorphic with constant less than 2. (See also [B, Corollary].)

As a corollary to Theorem 0.2 we obtain the result that if X is a $P_{1+\varepsilon}$ space for every $\varepsilon > 0$ then X is a P_1 space. This result was originally proved by

Lindenstrauss [L, Theorem 6.10] and we would like to thank him for suggesting that our methods might provide a simpler proof of this result.

Finally in Section 3 we construct a subspace X of $C(\omega^\omega)$ so that X^* is isometric to l_1 but X is not isomorphic to a complemented subspace of any $C(K)$ space, i.e., $\lambda(X) = \infty$. The construction is similar to that in [B–L] but the proof is simpler.

We shall use standard notation and terminology from Banach space theory. Unexplained terms and background material may be found in [L–T]. In particular $C(K)$ will denote the space of *real* valued continuous functions on a compact Hausdorff space K and we identify $C(K)^*$ with the finite regular (signed) Borel measures on K. For μ in $C(K)^*$ we use μ^+ and μ^- to denote its positive and negative parts, i.e., $\mu = \mu^+ - \mu^-$. If A is a Borel subset of K, $\mu_{|A}$ will denote the restriction of μ to A. Hence $\mu_{|A}(B) = \mu(A \cap B)$, for all Borel sets B. Closure and convergence in a dual space will be with respect to the w* topology unless otherwise noted.

§1. Efficient splittings

The proof of Proposition 0.1 requires that we construct splittings. Our first lemma contains the actual construction that we will need.

LEMMA 1.1. *Let ν, μ, μ_1 and μ_2 be finite measures on the measurable space (Ω, Σ) such that $\mu = \mu_1 + \mu_2$ and $\|\mu\| = \|\mu_1\| + \|\mu_2\|$. Let $\mathbf{P}$ be a finite partition of Ω into measurable sets such that μ has constant sign on each $A \in \mathbf{P}$. Then there are measures ν_1, ν_2, and η such that*

(i) $\nu = \nu_1 + \nu_2 + \eta$,

(ii) $\nu_j(A) = \mu_j(A)$ *for all* $A \in \mathbf{P}$ *and* $j = 1$ *and* 2,

(iii) $\|\nu_j\| \leqq \|\mu_j\|$ *for* $j = 1$ *and* 2,

(iv) $\|\eta\| \leqq \|\nu\| - \|\mu\| + 2\,\Sigma_{A\in\mathbf{P}}|\mu(A) - \nu(A)|$.

PROOF. Observe that because μ has constant sign on the sets in $\mathbf{P}$ and $\|\mu\| = \|\mu_1\| + \|\mu_2\|$, sign $\mu_{1|A}$ = sign $\mu_{2|A}$ = sign $\mu_{|A}$. For $j = 1$ and 2 and all A in $\mathbf{P}$, define

$$(*)\qquad \nu_{j|A} = \begin{cases} [\mu_j^+(A)/\nu^+(A)]\nu^+_{|A} & \text{if } \mu^+(A) \geqq 0 \text{ and } \nu^+(A) \neq 0, \\ -[\mu_j^-(A)/\nu^-(A)]\nu^-_{|A} & \text{if } \mu^-(A) > 0 \text{ and } \nu^-(A) \neq 0, \end{cases}$$

otherwise μ and ν have constant and opposite signs on A, and we put

(**) $$\nu_{j|A} = \mu_{j|A}.$$

Define $\eta = \nu - \nu_1 - \nu_2$.

It is easy to see that (i), (ii) and (iii) are satisfied. For (iv) note that for each $A \in \mathbf{P}$ at least one of $\mu^+(A)$ and $\mu^-(A)$ is zero. Therefore on sets A in $\mathbf{P}$ for which (*) is used and $\mu(A) \geqq 0$,

$$\begin{aligned}\| \eta_{|A} \| &= \| \nu^+_{|A} - [\mu_1^+(A)/\nu^+(A)]\nu^+_{|A} - [\mu_2^+(A)/\nu^+(A)]\nu^+_{|A} \| + \| \nu^-_{|A} \| \\ &= |\nu^+(A) - \mu^+(A)| + \| \nu^-_{|A} \| \\ &\leqq \| \nu_{|A} \| - \| \mu_{|A} \| + 2|\nu(A) - \mu(A)|.\end{aligned}$$

A similar calculation yields the same estimate if (*) is used and $\mu(A) < 0$. On the sets A for which (**) is used

$$\| \eta_{|A} \| = \| \nu_{|A} \| + \| \mu_{|A} \| = |\nu(A)| + |\mu(A)|$$

because both μ and ν have constant sign on A and the signs are opposite. Clearly

$$|\nu(A)| + |\mu(A)| \leqq \| \nu_{|A} \| - \| \mu_{|A} \| + 2|\nu(A) - \mu(A)|.$$

This proves (iv). ■

Proof of Proposition 0.1. For simplicity we shall assume in the proof that K is totally disconnected and that there are clopen sets H^+ and H^- such that μ is non-negative on H^+ and non-positive on H^-. The second assumption can be removed by noting that for every $\varepsilon > 0$ there is a closed set F such that $|\mu|(K \setminus F) < \varepsilon$ and $\mu_{|F}$ has a Hahn decomposition with clopen sets. Thus the proposition holds for $\mu_{|F}$, $\mu_{i|F}$, $i = 1, 2$, and $\mathscr{M} + (\mu_{|F} - \mu)$ and a simple diagonalization argument completes the proof. The proposition in the case that K is not totally disconnected can be proved by first composing with a projection as we do in the proof of Theorem 1.3.

Define $\mathscr{F}$ to be the net of all finite partitions of K generated by clopen subsets, i.e., every $\mathbf{P} \in \mathscr{F}$ is a finite collection of disjoint clopen sets $\{O_i\}$ such that $\bigcup O_i = K$. For each $\mathbf{P} \in \mathscr{F}$ let $\mathbf{P}'$ be the partition generated by $\mathbf{P}$ and $\mathscr{H} = \{H^+, H^-\}$, the Hahn decomposition of K relative to μ. Let $\nu^{\mathbf{P}} \in \mathscr{M}$ such that

$$|\nu^{\mathbf{P}}(A) - \mu(A)| < 1/|\mathbf{P}|^2 \qquad \text{for all } A \in \mathbf{P},$$

where $|\mathbf{P}|$ denotes the cardinality of $\mathbf{P}$. By Lemma 1.1 with $\nu = \nu^{\mathbf{P}}$ and the partition $\mathbf{P}'$ we can find $\nu_1^{\mathbf{P}}$, $\nu_2^{\mathbf{P}}$, and $\eta^{\mathbf{P}}$ in $C(K)^*$ such that

(i) $\nu^{\mathbf{P}} = \nu_1^{\mathbf{P}} + \nu_2^{\mathbf{P}} + \eta^{\mathbf{P}}$,

(ii) $\nu_j^{\mathbf{P}}(A) = \mu_j(A)$ for all $A \in \mathbf{P}'$ and $j = 1$ and 2,

(iii) $\| \nu_j^{\mathbf{P}} \| \leqq \| \mu_j \|$ for $j = 1$ and 2,

(iv) $\| \eta^{\mathbf{P}} \| \leqq \| \nu^{\mathbf{P}} \| - \| \mu \| + 2 \Sigma_{A \in \mathbf{P}'} | \mu(A) - \nu^{\mathbf{P}}(A) |$.

Because the topology of K is determined by its clopen sets, (ii) implies that $\nu_j^{\mathbf{P}} \to \mu_j$, $j = 1, 2$, and $\eta^{\mathbf{P}} \to 0$. From (iv) we get that

$$\lim(\| \nu^{\mathbf{P}} \| - \| \eta^{\mathbf{P}} \| - \| \mu \|) = 0. \qquad \blacksquare$$

In the proof of Theorem 1.3 we shall need the following simple perturbation result which will allow us to overcome small defects in the triangle inequality in the splitting $x^* = x_1^* + x_2^*$.

LEMMA 1.2. *Let μ, μ_1, and μ_2 be measures on the measurable space (Ω, Σ) with $\mu = \mu_1 + \mu_2$. Then there are measures ν_1 and ν_2 satisfying*

(i) $\mu = \nu_1 + \nu_2$,

(ii) $\| \mu \| = \| \nu_1 \| + \| \nu_2 \|$,

(iii) $\| \nu_i \| \leqq \| \mu_i \|$ *for* $i = 1$ *and* 2,

(iv) $\| \mu_1 - \nu_1 \| + \| \mu_2 - \nu_2 \| = \| \mu_1 \| + \| \mu_2 \| - \| \mu \|$.

PROOF. Let $\eta = |\mu_1| + |\mu_2|$. Clearly μ and the μ_i are absolutely continuous with respect to η. Let $f = d\mu/d\eta$ and $f_i = d\mu_i/d\eta$, for $i = 1$ and 2, in $L_1(\eta)$, and observe that it suffices to find g_1 and g_2 in $L_1(\eta)$ so that (i)–(iv) hold with f's and g's in place of the μ's and ν's, respectively.

Let $A = \{ f_1 f_2 \geqq 0 \}$, $B = \{ f_1 f_2 < 0$ and $|f_1| \leqq |f_2| \}$, and $C = \{ f_1 f_2 < 0$ and $|f_1| > |f_2| \}$ (here $\{ f > a \} = \{ \omega \in \Omega : f(\omega) > a \}$, etc.) and define

$$g_1(\omega) = \begin{cases} f_1(\omega) & \text{if } \omega \in A, \\ 0 & \text{if } \omega \in B, \\ f_1(\omega) + f_2(\omega) & \text{if } \omega \in C, \end{cases}$$

that is,

$$g_1 = f_1 1_{A \cup C} + f_2 1_C.$$

Define

$$g_2 = f_1 1_B + f_2 1_{A \cup B}.$$

Then $f = g_1 + g_2$, $g_1 g_2 \geqq 0$ everywhere, and thus $\| f \| = \| g_1 + g_2 \| = \| g_1 \| + \| g_2 \|$. Clearly $\| g_1 \| \leqq \| f_i \|$ for $i = 1$ and 2, and

$$\begin{aligned} \| f_1 - g_1 \| + \| f_2 - g_2 \| &= 2 \left[\int_B |f_1| \, d\eta + \int_C |f_2| \, d\eta \right] \\ &= \| f_1 \| + \| f_2 \| - \| f \|. \qquad \blacksquare \end{aligned}$$

We are now ready for the main result of this section.

THEOREM 1.3. *Let X be a Banach space with $\lambda(X) < 1 + \delta$ and let $\mathcal{M}$ be a subset of the unit ball of X^*. Suppose that x^* is a w*-limit point of $\mathcal{M}$ and that there are x_1^* and x_2^* in X^* such that $x^* = x_1^* + x_2^*$ and $\|x_1^*\| + \|x_2^*\| = \|x^*\|$. Then there is a net (f^α) in $\mathcal{M}$ and for each α, elements f_1^α, f_2^α, and g^α of X^* such that*

(i) $f^\alpha = f_1^\alpha + f_2^\alpha + g^\alpha$,

(ii) $f^\alpha \to x^*, f_1^\alpha \to x_1^*, f_2^\alpha \to x_2^*$ *and* $g^\alpha \to 0$,

(iii) $\|f_i^\alpha\| \leqq \|x_i^*\| + 2\delta$ *for all* α *and* $i = 1$ *and* 2,

(iv) $\lim_\alpha(\|g^\alpha\| + \|x^*\| - \|f^\alpha\|) < \delta$.

PROOF. By Milutin's Theorem [M] in the metrizable case and Ditor's [D] in the general case, every $C(K)$ space is norm one complemented in a $C(S)$ space, with S totally disconnected. We can thus assume that $X \subset C(K)$, where K is totally disconnected and that $Q: C(K) \to X$ is a projection with $\|Q\| < 1 + \delta$.

Let $\mu = Q^*x^*$ and $\mu_i = Q^*x_i^*$, for $i = 1$ and 2, then

$$\|\mu_1\| + \|\mu_2\| < (1+\delta)(\|x_1^*\| + \|x_2^*\|)$$

$$= (1+\delta)\|x^*\| \leqq \|\mu\| + \delta\|x^*\| \leqq \|\mu\| + \delta.$$

By Lemma 1.2 there are $\nu_i \in C(K)^*$, $i = 1, 2$, such that $\mu = \nu_1 + \nu_2$,

$$(*) \qquad \|\mu\| = \|\nu_1\| + \|\nu_2\|,$$

$$\|\nu_i\| \leqq \|\mu_i\|,$$

and

$$(**) \qquad \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\| = \|\mu_1\| + \|\mu_2\| - \|\mu\| < \delta.$$

By Proposition 0.1 there are measures $\nu^\alpha \in Q^*\mathcal{M}$, ν_i^α and η^α in $C(K)^*$ such that

(1) $\nu^\alpha = \nu_1^\alpha + \nu_2^\alpha + \eta^\alpha$,

(2) $\nu^\alpha \to \mu$, $\nu_1^\alpha \to \nu_1$, $\nu_2^\alpha \to \nu_2$ and $\eta^\alpha \to 0$,

(3) $\|\nu_i^\alpha\| \leqq \|\nu_i\|$ for all α and $i = 1$ and 2,

(4) $\lim_\alpha(\|\nu^\alpha\| - \|\eta^\alpha\| - \|\mu\|) = 0$.

Let $\gamma_i^\alpha = \nu_i^\alpha + \mu_i - \nu_i$ and note that by (2), $\gamma_i^\alpha \to \mu_i$. Define f^α, f_i^α, and g^α to be the restrictions to X of ν^α, γ_i^α, and η^α, respectively. Then $f_i^\alpha \to x_i^*$ and $g^\alpha \to 0$. As $\lim \|\nu_i^\alpha\| = \|\nu_i\|$ by (2) and (3), we obtain that

$$\begin{aligned}\lim \| f_i^\alpha \| \leqq \lim \| \gamma_i^\alpha \| &\leqq \| \mu_i - \nu_i \| + \lim \| \nu_i^\alpha \| \\ &= \| \mu_i - \nu_i \| + \| \nu_i \| \\ &\leqq \| x_i^* \| + 2\delta,\end{aligned}$$

(because by (*), $\| \nu_i \| \leqq \| \mu_i \| \leqq (1+\delta) \| x_i^* \| \leqq \| x_i^* \| + \delta$ and by (**) $\| \mu_i - \nu_i \| < \delta$).

Finally

$$\begin{aligned}&\lim(\| g^\alpha \| + \| x^* \| - \| f^\alpha \|) \\ &\leqq \lim(\| \eta^\alpha \| + \| \mu \| - (1+\delta)^{-1} \| \nu^\alpha \|) \\ &= \lim(\| \eta^\alpha \| + \| \mu \| - \| \nu^\alpha \|) + \lim \delta(1+\delta)^{-1} \| \nu^\alpha \| < \delta,\end{aligned}$$

by (4) and because $\| \nu^\alpha \| \leqq 1 + \delta$. ■

In Section 3 we shall work with separable spaces X only. In this case bounded subsets of X^* are metrizable in the w* topology, and instead of nets we shall be able to use sequences. For convenience we now formulate the "efficient splitting" phenomena in separable $C(K)$ spaces exactly as it will be used in Section 3. The proof is essentially the same as that above so we omit it.

PROPOSITION 1.4. *Let K be a compact metric space and let $\varepsilon > 0$. Suppose that $(\nu^n)_{n \in \mathbf{N}}$, μ, μ_1, and μ_2 are measures on K such that $\mu = (\mu_1 + \mu_2)/2$, $\nu^n \to \mu$, and $(\| \mu_1 \| + \| \mu_2 \|)/2 \leqq \| \mu \| + 2\varepsilon$. Then there is a subsequence $(\nu^n)_{n \in M}$ and for each $n \in M$ measures ν_1^n and ν_2^n such that*

(i) $\nu^n = (\nu_1^n + \nu_2^n)/2$,

and for $i = 1$ and 2,

(ii) $\nu_i^n \to \mu_i$,

(iii) $\lim_n \| \nu_i^n \| \leqq \| \mu_i \| + 4\varepsilon + \lim_n(\| \nu^n \| - \| \mu \|)$.

§2. The proof of Theorem 0.2

Norm one complemented subspaces of $C(K)$ spaces were characterized by Lindenstrauss and Wulbert [L–W] as the $C_\sigma(K)$ spaces. They also proved that an L_1-predual is a $C_\sigma(K)$ space, with σ fixed point free if and only if $\mathscr{E} = \operatorname{Ext} B_{X^*}$ is w* closed. The following proposition gives a similar characterization of general $C_\sigma(K)$ spaces. The proposition is a simple variation of [L–W, Theorem 1] and must be known, but we do not know of a suitable reference. We shall only give a sketch of the argument using the same notation as in [L–W].

PROPOSITION 2.1. *Let X be an L_1-predual and let $\mathscr{E} = \operatorname{Ext} B_{X^*}$. If $\bar{\mathscr{E}}^{w^*} \subset \mathscr{E} \cup \{0\}$, then X is isometric to a $C_\sigma(K)$ space, i.e., to a norm one complemented subspace of a $C(K)$ space.*

SKETCH OF PROOF. Let $K = \bar{\mathscr{E}}^{w^*}$ and let P_α be a net of finite rank projections from $C(K)$ into X such that $P_\alpha x \to x$ for all $x \in X$ and $\| P_\alpha \| \to 1$. Let $T_\alpha = P_\alpha / \| P_\alpha \|$ for all α. We shall show that $T_\alpha f \xrightarrow{w} f$ for all $f \in C_\sigma(K)$, where $\sigma(k) = -k$. Fix $f \in C_\sigma(K)$ and $\mu \in C_\sigma(K)^*$. We identify μ with an odd regular Borel measure on K. (Here odd means that $\mu(\sigma A) = -\mu(A)$ for every Borel set A such that $\sigma(A) \cap A = \varnothing$.) Given $\varepsilon > 0$ let $\mathscr{U}$ be an open neighborhood of $0 \in K$ so that $|\mu|(\mathscr{U}) < \varepsilon$. Then $\mathscr{E} \setminus \mathscr{U}$ is a w*-compact subset of $K = \mathrm{cb}(C_\sigma(K))$, the generalized Choquet boundary of $C_\sigma(K)$. By [W, Lemma 1 (ii)], $T_\alpha f(k) \to f(k)$ uniformly on $\mathscr{E} \setminus \mathscr{U}$. Thus

$$|\langle f - T_\alpha f, \mu \rangle| = \left| \int (f - T_\alpha f) d\mu \right|$$

$$\leqq \int_{\mathscr{U}} |T_\alpha f - f| \, d|\mu| + \int_{\mathscr{E} \setminus \mathscr{U}} |T_\alpha f - f| \, d|\mu|.$$

The first term is at most $2\varepsilon \| f \|$ and the second converges to 0, as noted above. ∎

The next lemma loosely speaking says that the unit ball of an $L_1(\mu)$ space has a uniform modulus of convexity at the extreme points.

LEMMA 2.2. *Let e be an extreme point of the unit ball of an $L_1(\mu)$ space and let $\delta > 0$. If $e = f + g$, with $\| f \| + \| g \| < 1 + \delta$, then there are real numbers s and t such that*

$$s + t = 1 \quad \textit{and} \quad \| f - se \| + \| g - te \| < \delta.$$

PROOF. By multiplying by -1 if necessary, we can assume that e is the indicator function of an atom of μ. Let $f = se + h_1$ and $g = te + h_2$ where h_1 and h_2 are 0 on the atom. Clearly $s + t = 1$ and

$$|s| + |t| + \| h_1 \| + \| h_2 \| < 1 + \delta.$$

Thus

$$\delta > \| h_1 \| + \| h_2 \| = \| f - se \| + \| g - te \|. \qquad ∎$$

We are now ready for the

PROOF OF THEOREM 0.2. It is well known that X^* is isometric to an $L_1(\mu)$ space. (See [L–T, p. 157].) Thus by Proposition 2.1 we need only show that

$\bar{\mathscr{E}}^{w^*} \subset \mathscr{E} \cup \{0\}$, where $\mathscr{E} = \operatorname{Ext} B_{X^*}$. Fix $x^* \in \bar{\mathscr{E}}^{w^*} \setminus \{0\}$. We need to show that $x^* \in \mathscr{E}$.

Suppose that $x^* = x_1^* + x_2^*$ where $\| x^* \| = \| x_1^* \| + \| x_2^* \|$ and let $\varepsilon > 0$. By Theorem 1.3 there is net of extreme points (e_α) converging to x^* and there is a splitting of (e_α) in X^*, (f_1^α), (f_2^α), and (g^α) such that

$$e_\alpha = f_1^\alpha + f_2^\alpha + g^\alpha, \quad f_i^\alpha \to x_i^*, \quad \| f_i^\alpha \| \leqq \| x_i^* \| + \varepsilon, \qquad i = 1, 2,$$

and

$$\lim(\| g^\alpha \| + \| f_1^\alpha \| + \| f_2^\alpha \| - \| e_\alpha \|) < \varepsilon.$$

By Lemma 2.2 for each α there are real numbers t_1^α and t_2^α such that

$$t_1^\alpha + t_2^\alpha = 1$$

and

$$\| f_1^\alpha - t_1^\alpha e_\alpha \| + \| f_2^\alpha + g^\alpha - t_2^\alpha e_\alpha \| < \varepsilon.$$

By passing to a subnet we can assume that $t_i^\alpha \to t_i(\varepsilon)$, $i = 1$ and 2, and passing to the limit, we get that

$$\| x_1^* - t_1(\varepsilon) x^* \| + \| x_2^* - t_2(\varepsilon) x^* \| \leqq \varepsilon.$$

Because ε is arbitrary both x_1^* and x_2^* must be multiples of x^* and therefore x^* is a multiple of an extreme point.

To see that x^* is an extreme point let $x_1^* = x^*$ and $x_2^* = 0$ above. Then we must have that $t_1(\varepsilon) \geqq 1 - \varepsilon / \| x^* \|$ and hence for large α, $t_1^\alpha > 1 - 2\varepsilon / \| x^* \|$. Consequently

$$\| x^* \| = \| x_1^* \| \geqq \| f_1^\alpha \| - \varepsilon \geqq t_1^\alpha - 2\varepsilon > 1 - 2\varepsilon / \| x^* \| - 2\varepsilon,$$

that is, $\| x^* \| = 1$. ■

Corollary 2.3. *If X is a $P_{1+\varepsilon}$ space for every $\varepsilon > 0$, then X is a P_1 space.*

Proof. By Theorem 0.2, X is a $C_\sigma(K)$ space. We shall show that a $C_\sigma(K)$ space is in fact a $C(S)$ space if it is a P_λ space for $\lambda < 2$. The corollary then follows from a result of Amir [A] and Isbell and Semadeni [I–S]. By Zorn's lemma there is a maximal open set $\mathscr{U}$ which is disjoint from $\sigma(\mathscr{U})$. Let $h = 1_{\mathscr{U}} - 1_{\sigma(\mathscr{U})}$ and $f = Ph$ where P is a projection from $l_\infty(K)$ onto X and $\| P \| < t < 2$. Then $f(k) > 2 - t$ for all $k \in \mathscr{U}$. Indeed, for $k \in \mathscr{U}$, let $g \in X$ such that $\operatorname{supp} g \subset \mathscr{U} \cup \sigma(\mathscr{U})$, $g \geqq 0$ on $\mathscr{U}$ and $g(k) = 1$. Thus $\| h - 2g \| = 1$ and $\| P(h - 2g) \| = \| f - 2g \| < t$. Hence

$$|f(k)-2g(k)| = |f(k)-2| < t,$$

as claimed.

By the maximality of $\mathcal{U}$ it is equal to the set where f is positive, and we have shown above that $f \geqq 2-t > 0$ on $\mathcal{U}$, therefore $\mathcal{U}$ is clopen. Also $K \setminus (\mathcal{U} \cup \sigma(\mathcal{U}))$ is clopen and again by the maximality of $\mathcal{U}$, this set must be the intersection of the zero sets of all $x \in X$. It follows that the map R from $C(K)$ into $C(\mathcal{U})$ defined by $Rx = x_{|\mathcal{U}}$ is an isometry from X onto $C(\mathcal{U})$. ■

§3. An uncomplemented l_1-predual

In this section we describe a slight modification of the l_1-predual constructed by Benyamini and Lindenstrauss [B–L], and use the idea of "efficient splitting" to give a rather simple proof that it is not complemented in any $C(K)$ space. For each $n \in \mathbf{N}$ we shall construct an l_1-predual X_n such that $\lambda(X_n) \to \infty$. We then take $X = (\Sigma \oplus X_n)_{c_0}$ as the desired example. The space X_n will be described as a certain subspace of $C(K_n)$ for a suitably constructed space K_n. We begin with some topological considerations and the construction of K_n.

Let K be a compact metric space. We denote by $I(K)$ the set of isolated points of K, and by K' the first derived set, i.e., $K' = K \setminus I(K)$. If S is another compact metric space, we define a new space $K(S)$, *the substitution of S into K*, by replacing each isolated point $a \in I(K)$ by a copy of S. More precisely, $K(S)$ is the disjoint union $K' \cup (\bigcup \{S^a : a \in I(K)\})$, where each S^a is homeomorphic to S and is clopen in $K(S)$. A base for the topology at a point $k \in K' \subset K(S)$ is given by sets of the form $(K' \cap G) \cup (\bigcup \{S^a : a \in I(K) \cap G\})$, where G is an open subset of K containing k. It is easy to check that $K(S)$ is a compact metric space, and that if $x_n \in S^{a_n}$ for different a_n's in $I(K)$, then $x_n \to x$ in $K(S)$ if and only if $x \in K'$ and $a_n \to x$ in K.

We now define the spaces K_n. Let K_1 be the disjoint union of three convergent sequences. Inductively we define $K_{n+1} = K_1(K_n)$. It is easy to see that K_n is homeomorphic to the space of ordinals less than or equal to $\omega^n \cdot 3$ in the order topology, and that $K_{n+m} = K_n(K_m)$ for all $n, m \geqq 1$. We denote by a_1, a_2, a_3 the three limit points of K_1, as well as their copies in K_n in the representation $K_1(K_{n-1})$, i.e., $\{a_1, a_2, a_3\}$ is the nth derived set of K_n.

Finally we inductively define the spaces X_n as follows:

$$X_1 = \{f \in C(K_1) : f(a_1) = (f(a_2) + f(a_3))/2\}.$$

Assume $X_n \subset C(K_n)$ has been defined. For each isolated point $a \in K_1$, there is a copy K_n^a of K_n in $K_1(K_n)$. Let X_n^a be the natural copy of X_n in $C(K_n^a)$. We then define

$$X_{n+1} = \{f \in C(K_{n+1}) : f(a_1) = (f(a_2) + f(a_3))/2 \text{ and } f_{|K_n^a} \in X_n^a \text{ for all } a \in I(K_1)\}.$$

We shall consider K_n as a subset of X_n^* by identifying a point $k \in K_n$ with the functional $\delta(k)$, evaluation at k. It can be shown that X_n^* is isometric to l_1 with basis drawn from these point evaluations. This follows inductively from the fact that each $x^* \in X_{n+1}^*$ has a unique representation

$$x^* = \alpha_2 \delta(a_2) + \alpha_3 \delta(a_3) + \sum x_a^*,$$

where the sum is over all $a \in I(K_1)$ and $x_a^* \in (X_n^a)^*$. Moroever $\| x^* \| = |\alpha_2| + |\alpha_3| + \Sigma \| x_a^* \|$ and thus $X_{n+1}^* = (\mathbf{R} \oplus \mathbf{R} \oplus \Sigma \oplus (X_n^a)^*)_{l_1}$.

In order to prove that $\lambda(X_n) \to \infty$ we will need to pass from a given complemented copy of X_n in some $C(K)$ to a more nicely positioned subspace isomorphic to X_m for some $m < n$. Assume that $X_n \subset C(S)$ and $P : C(S) \to X_n$ is a projection. Consider the measures $\{P^* \delta(k) : k \in K_n\}$. In order to use "efficient splittings" we shall need all of these measures to have essentially the same norm. This is achieved in the next lemma by passing from X_n to some natural copy of X_m in X_n for suitable $m < n$.

Lemma 3.1. *Let X_n, $C(S)$ and P be as above and fix $\varepsilon > 0$. If $n > m \| P \| / 2\varepsilon$, then there is a constant C and a complemented copy Y of X_m in X_n with projection Q such that*

$$C - \varepsilon \leqq \| (QP)^* \delta(k) \| \leqq C + \varepsilon$$

for all $k \in K_m \subset X_m^*$.

Proof. Represent K_n as $K_{n-m}(K_m)$, let $C' = \| P \| - 2\varepsilon$ and consider two possibilities:

(i) There is an $a \in I(K_{n-m})$ such that $\| P^* \delta(k) \| \geqq C'$ for all $k \in K_m^a$. We then take $C = C' + \varepsilon$ and X_m to be the set of all functions in X_n with support contained in K_m^a and define the projection Q by $QF = f 1_{K_m^a}$. This is the required subspace.

(ii) If for each $a \in I(K_{n-m})$ there is a $k^a \in K_m^a$ with $\| P^* \delta(k^a) \| < C'$, consider the set $B_{n-m} = K'_{n-m} \cup \{k^a : a \in I(K_{n-m})\}$. B_{n-m} is homeomorphic to K_{n-m} and $\{k^a : a \in I(K_{n-m})\}$ is dense in B_{n-m}. Because $\| P^* \delta(k) \| < C'$ on this dense set and the map $k \to \| P^* \delta(k) \|$ is lower semi-continuous, it follows that $\| P^* \delta(k) \| \leqq C'$ for all $k \in B_{n-m}$. This gives us a natural copy of X_{n-m} in X_n, namely

$$Z = \{ f \in C(K_n) : f \text{ is constant on } K_m^a \text{ for each } a \in I(K_{n-m})\},$$

and there is a norm one projection onto Z given by

$$Qf(k) = \begin{cases} f(k) & \text{if } k \in K'_{n-m} \\ f(k^a) & \text{if } k \in K_m^a \text{ for some } a \in I(K_{n-m}) \end{cases}$$

for which $\| (QP)^*\delta(k) \| \leqq C'$ for all $k \in B_{n-m}$.

If (ii) occurs we repeat the argument with $C' = \| P \| - 2(r+1)\varepsilon$, B_{n-rm} in place of K_n, and X_{n-rm} in place of X_n for $r = 1, 2, \ldots$. We must succeed, i.e., case (i) must occur, for some $r < \| P \| / 2\varepsilon$. ∎

We are now ready to prove that $\lambda(X_n) \to \infty$. Assume not and let $\lambda \geqq \lambda(X_n)$ for all n and let $\varepsilon > 0$. By Lemma 3.1 for each n there is an embedding $X_n \subset C(S)$ with a projection P from $C(S)$ onto X_n of norm at most λ and a constant $C \leqq \lambda$ such that $C - \varepsilon \leqq \| P^*\delta(k) \| \leqq C + \varepsilon$ for all $k \in K_n$. Fix n. We shall prove by induction the following:

CLAIM. There are measures (μ_i) on $C(S)$ satisfying

(i) $\mu_i \to P^*\delta(a_2)$ or $\mu_i \to P^*\delta(a_3)$,

(ii) $\| \mu_i \| \leqq C + 8n\varepsilon$,

(iii) $\| \mu_{i|X_n} \| \geqq (1-\varepsilon)(n+1)$.

Because $(P^*\mu_i)_{|X_n} = \mu_{i|X_n}$, combining (ii) and (iii) and using the fact that $C \leqq \lambda$, yields

$$(1-\varepsilon)n \leqq \| \mu_{i|X_n} \| = \| (P^*\mu_i)_{|X_n} \| \leqq \| P^*\mu_i \| \leqq \| P^* \| \, \| \mu_i \| \leqq \lambda(\lambda + 8n\varepsilon).$$

If $\varepsilon < 1/16\lambda$ and $n > 16\lambda^3/(8\lambda - 1)$, this is a contradiction.

PROOF OF THE CLAIM. Assume that $n = 1$. For $i = 1, 2, 3$, let $t_{i,j} \to a_i$ be the three convergent sequences whose union is K_1. Thus

$$P^*\delta(t_{1j}) \to P^*\delta(a_1) = (P^*\delta(a_2) + P^*\delta(a_3))/2$$

and

$$(\| P^*\delta(a_2) \| + \| P^*\delta(a_3) \|)/2 \leqq C + \varepsilon \leqq \| P^*\delta(a_1) \| + 2\varepsilon.$$

By Proposition 1.4 there is a subsequence (which, for simplicity of notation, we assume is the whole sequence), and for each j, a splitting $P^*\delta(t_{1,j}) = (\mu_2^j + \mu_3^j)/2$ such that $\mu_i^j \to P^*\delta(a_i)$ for $i = 2, 3$ and

$$\lim \| \mu_i^j \| \leqq \| P^*\delta(a_i) \| + 4\varepsilon + \lim \| P^*\delta(t_{1,j}) \| - \| P^*\delta(a_1) \|$$
$$\leqq C + \varepsilon + 4\varepsilon + C + \varepsilon - (C - \varepsilon) = C + 7\varepsilon.$$

By passing to a tail of the sequence we obtain that $\| \mu_i^j \| \leqq C + 8\varepsilon$ and so (i) and (ii) hold for both sequences μ_2^j and μ_3^j. We now estimate $\| \mu_{i|X_1}^j \|$. Define f^j and f in X_1 by

$$f^j = \mathbf{1}_{\{t_{1,j}\}} \quad \text{and} \quad f = \mathbf{1}_{\{a_2\} \cup \{t_{2,j} : j \in \mathbf{N}\}} - \mathbf{1}_{\{a_3\} \cup \{t_{3,j} : j \in \mathbf{N}\}}.$$

As $1 = \langle f^j, P^*\delta(t_{1,j}) \rangle = (\mu_2^j(f^j) + \mu_3^j(f^j))/2$, it follows that either $\mu_2^j(f^j) \geqq 1$ or $\mu_3^j(f^j) \geqq 1$ for infinitely many j. Suppose that this happens for μ_2^j. By passing to a further subsequence, we can assume that $\mu_2^j(f^j) \geqq 1$ for all j. Now $\mu_2^j \to P^*\delta(a_2)$, hence

$$\mu_2^j(f) \to \langle f, P^*\delta(a_2) \rangle = 1 \quad \text{and} \quad \| f^j + f \| = 1 \qquad \text{for all } j.$$

Therefore $\| \mu_{2|X_1}^j \| \geqq \mu_2^j(f^j + f) \geqq 2(1 - \varepsilon)$ for sufficiently large j, and (μ_2^j) satisfies (iii) as well as (i) and (ii). If $\mu_3^j(f^j) \geqq 1$ for infinitely many j, we use $f^j - f$ in place of $f^j + f$ and proceed similarly.

Now we will consider the induction step.

Write K_{n+1} as the union $\{a_1, a_2, a_3\} \cup (\bigcup K_n^{i,j})$ where each $K_n^{i,j}$ is the copy of K_n which is substituted for $t_{i,j}$ in K_1 in the representation $K_{n+1} = K_1(K_n)$. For each i and j let $X_n^{i,j}$ be the natural copy of X_n supported on $K_n^{i,j}$ and $P^{i,j}$ be the natural projection onto $X_n^{i,j}$, i.e.,

$$X_n^{i,j} = \{ f \in X_{n+1} : f(k) = 0 \text{ for } k \notin K_n^{i,j} \} \quad \text{and} \quad P^{i,j}f = f1_{K_n^{i,j}}.$$

Then $(P^{i,j})^*\delta(k) = \delta(k)$ for $k \in K_n^{i,j}$. Let $a_1^{i,j}$, $a_2^{i,j}$, and $a_3^{i,j}$ be the points in the last non-empty derived set of $K_n^{i,j}$. By the induction hypothesis, for each j there is a sequence of measures $(\mu_k^j)_{k \in \mathbf{N}}$ on S such that

(i) $\mu_k^j \to (P^{1,j}P)^*\delta(a_2^{1,j}) = P^*\delta(a_2^{1,j})$ or $\mu_k^j \to P^*\delta(a_3^{1,j})$,

(ii) $\| \mu_k^j \| \leqq C + 8n\varepsilon$,

(iii) $\| \mu_{k|X_n^{1,j}}^j \| \geqq (1 - \varepsilon)(n + 1)$.

Both of the sequences $P^*\delta(a_2^{1,j})$ and $P^*\delta(a_3^{1,j})$ converge to $P^*\delta(a_1)$, and all measures considered are in a compact metrizable space, namely, the ball of radius $\lambda + 8n\varepsilon$ in $C(S)^*$. Hence we can diagonalize and choose for each j an index $k(j)$ such that $\nu^j = \mu_{k(j)}^j \to P^*\delta(a_1)$.

As in the case $n = 1$, we use Proposition 1.4 and split ν^j as $\nu^j = (\mu_2^j + \mu_3^j)/2$ so that for $i = 2, 3$

$$\mu_i^j \to P^*\delta(a_i)$$

and

$$\lim \| \mu_i^j \| \leqq \| P^*\delta(a_i) \| + 4\varepsilon + \lim \| \nu^j \| - \| P^*\delta(a_1) \|$$
$$\leqq C + \varepsilon + 4\varepsilon + C + 8n\varepsilon - C + \varepsilon < C + 8(n+1)\varepsilon.$$

Finally we estimate $\| \mu^j_{i|X_{n+1}} \|$. For each j fix $f^j \in X_n^{1,j} \subset X_{n+1}$, with $\| f^j \| = 1$ such that

$$\nu^j(f^j) \geqq \| \nu^j_{|X_n^{1,j}} \| - \varepsilon/2 \geqq (n+1)(1-\varepsilon) - \varepsilon/2.$$

As $\nu^j(f^j) = (\mu_2^j(f^j) + \mu_3^j(f^j))/2$, either $\mu_2^j(f^j) \geqq (n+1)(1-\varepsilon) - \varepsilon/2$ or $\mu_3^j(f^j) \geqq (n+1)(1-\varepsilon) - \varepsilon/2$ for infinitely many j's. Suppose this happens for μ_2^j. By passing to a subsequence we may assume that

$$\mu_2^j(f^j) \geqq (n+1)(1-\varepsilon) - \varepsilon/2 \qquad \text{for all } j.$$

Let $f \in X_{n+1}$ be defined by

$$f = \mathbf{1}_{\{a_2\} \cup (\cup K_n^{2,j})} - \mathbf{1}_{\{a_3\} \cup (\cup K_n^{3,j})}.$$

Then $\| f^j \pm f \| = 1$ and because $\mu_2^j \to P^*\delta(a_2)$, $\mu_2^j(f) \to \langle f, P^*\delta(a_2) \rangle = 1$. Thus

$$\lim \| \mu^j_{2|X_{n+1}} \| \geqq \lim \mu_2^j(f^j + f) \geqq (n+1)(1-\varepsilon) - \varepsilon/2 + 1,$$

and

$$\| \mu^j_{2|X_{n+1}} \| \geqq (n+2)(1-\varepsilon)$$

provided j is large enough. ■

REMARKS. Here and in [B–L] three convergent sequences are used as the basic building block of the example. However, in the proof we seem only to use the sequence converging to a_1 and so it might seem that a similar but simpler construction could also be used to give an example with the same properties. Suppose we try the construction with one sequence.

Let $M_1 = \{t_i\} \cup \{a_1, a_2, a_3\}$ where $t_i \to a_1$ and a_2 and a_3 are isolated points; we can inductively define M_{n+1} as the "restricted" substitution of M_n into M_1, i.e., each t_i is replaced by a copy M_n^i of M_n but we leave a_2 and a_3 as isolated points. Then let

$$Y_1 = \{ f \in C(M_1) : f(a_1) = (f(a_2) + f(a_3))/2 \}$$

and inductively define

$$Y_{n+1} = \{ f \in C(M_{n+1}) : f(a_1) = (f(a_2) + f(a_3))/2 \text{ and } f_{|M_n^i} \in Y_n^i \text{ for all } i \}$$

where Y_n^i is the copy of Y_n supported on M_n^i.

If $P : C(S) \to Y_n$ is a projection for which $\| P^*\delta(m) \|$ is essentially constant for all $m \in M_n$, the proof given above shows that $\| P^* \| \sim m + 1$. The two other convergent sequences used in the construction of X_n were needed precisely to force $\| P^*\delta(k) \|$ to be essentially constant. Because those sequences are not present in the construction of Y_n we get that Y_n is 3-isomorphic to $C(\omega^n)$! We sketch the proof below.

Let $S_1 = \{t_i\} \cup \{a_1, a_2\} \subset M_1$ and if S_n has been defined, let S_n^i denote the copy of S_n in M_n^i and define $S_{n+1} = (\bigcup S_n^i) \cup \{a_1, a_2\} \subset M_{n+1}$. It is easy to see that S_n is homeomorphic to ω^n. Moreover the restriction map $R_n : C(M_n) \to C(S_n)$ defined by $R_n f = f_{|S_n}$ is a 3-isomorphism of Y_n onto $C(S_n)$. Indeed, $\| R_n \| = 1$ and there is a unique extension of an element of $C(S_n)$ to M_n which is in Y_n. Precisely, define $E_1 : C(S_1) \to Y_1$ by

$$E_1 f(m) = \begin{cases} f(m) & \text{if } m \in S_1 \\ 2f(a_1) - f(a_2) & \text{if } m = a_3 \end{cases}$$

and inductively define $E_{n+1} : C(S_{n+1}) \to Y_{n+1}$ by

$$E_{n+1} f(m) = \begin{cases} f(m) & \text{if } m = a_1 \text{ or } a_2 \\ 2f(a_1) - f(a_2) & \text{if } m = a_3 \\ E_n^i f(m) & \text{if } m \in M_n^i \text{ for some } i \end{cases}$$

where E_n^i is E_n but from $C(S_n^i)$ onto Y_n^i. It is not hard to check that E_n maps into Y_n, has norm at most 3, and is a right inverse for R_n.

Let us make one other observation about Y_n. By [B–L, Lemma] the study of $\lambda(X)$ was reduced to the study of projections onto isometric embeddings of X into $C(K)$. Dually one could ask if it could be reduced to the study of 1-quotient maps, i.e., if $\lambda(X) < \infty$, is there a compact Hausdorff space K and a 1-quotient map q from $C(K)$ onto X which admits a right inverse $T : X \to C(K)$ with $\| T \| \leqq \lambda(X)$? Unfortunately this is not always possible. While $\lambda(Y_n) \leqq 3$, if q is a η-quotient map from $C(K)$ onto Y_n with $\eta < 2$, an argument similar to the proof that $\lambda(X_n) \to \infty$ shows that any right inverse T of q will have $\| T \| \geqq cn$ for some constant c independent of n.

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